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THE FINANCIAL APPLICATIONS OF RANDOM CONTROL PROBLEM IN CONTINUOUS TIME

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ABSTRACT

The aim of the present paper is to find the value function and optimal control for controlled problem described by continuous time model

KEYWORDS: control problem, value function, random differential equation.

INTRODUCTION

The main goal of any investor is to gain maximum profit to his investment. The control problems and random control problems help the investors to realize their goals, this why this types of problems arising in many financial and economical applications of random control theory. The formulations of these types of problems depend on the nature of the problem itself that is either to maximize the profit or minimize the coast according to the requirements of the problem.

Controllability is one of the fundamental concept in mathematical control theory and plays an important role both in deterministic and random control theory see [1] , [2]. There are many different definitions of controllability, both for linear and nonlinear dynamical systems which depend on the class of dynamical control systems and the set of admissible controls see [3] , [4].

First let us consider a complete filtered probability space $(\Omega, F, \{F\}_{t>0}, P)$ where Ω is the set all possible outcomes of any random experiment, F represent the set of possible events which are sigma algebra, $\{F\}_{t \in T}$ is the filtration. We interpret $\{F\}_{t \in T}$ as representing the flow of information over time , with F_t being the information available at time t and P is the true or physical probability measure.

We say that the probability space $(\Omega, F, \{F\}_{t>0}, P)$ satisfying the usual conditions or usual hypotheses if the following conditions are met

- The (Ω, F, P) is complete.
- The σ – algebra F_t contain all the sets in F of zero probability.
- The filtration $\{F\}_{t \in T}$ is right continuous.

On this space we will define the following concepts:

Random Process: Random process indexed by T is collection of random variable defined by the map

$$X : \Omega \times T \rightarrow R^n \text{ such that } \forall t \in T, \omega \rightarrow X(\omega, t) \text{ is measurable}$$

Sample path: for fixed $\omega \in \Omega$, the sample path of the random process is the map $t \rightarrow X(t, \omega)$.

A process X_t is said to be continuous time random process if its sample path is a continuous function otherwise X_t is called a discontinuous random process (or process with jump component).

Wiener process: On the time interval $[0, \infty]$, a Wiener process $W(t, \omega) = W_t$ (Brownian motion) is a continuous random process with values in \mathbb{R} such that the following conditions hold:

- 1) $W_0 = 0$
- 2) For $0 \leq s \leq t \leq T$, $W_t - W_s$ has a normal distribution $N(0, t-s)$ with mean zero and variance $t-s$
- 3) independent increments, for $0 \leq s' < t' \leq s \leq t \leq T$, $W_t - W_s$ is independent of $W_{t'} - W_{s'}$.

Wiener process is a fundamental example of a random process and is of particular importance both in theory and in applications.

Stopping times: A random variable τ with values in $[0, \infty]$ is an $\{F_t\}$ -stopping time if $\{\tau \leq t\} \in F_t$, $\forall t \geq 0$. Stopping time is often defined by a stopping rule or a mechanism for deciding whether to continue or to stop a process.

Random differential equation is a differential equation in which one or more of the terms is a random process, resulting in a solution which is itself a random process.

Geometric Brownian motion

A random process X_t is said to follow a Geometric Brownian motion if it satisfies the following stochastic differential equation:

$$dX_t = r X_t dt + \sigma X_t dW_t, \quad t \geq 0$$

Where W_t is a Wiener process and r (percentage drift) and σ (the percentage volatility) are constants. The analytical solution of this geometric Brownian motion is given by:

$$X_t = X_0 e^{\left(r - \frac{\sigma^2}{2}\right)t} + \sigma W_t$$

Dimensional Ito formula: Ito's formula is the fundamental theorem of random calculus, just as one speaks of the fundamental theorem of ordinary calculus.

Let X_t be i -dimensional Ito processes given by: $dX_t = u(t) dt + v(t) dB_t$. Let

$g(x, t) \in C^2([0, \infty) \times \mathbb{R}^i)$, then $Y_t = g(t, X_t)$ is again an Ito process and

$$dY_t = \frac{\partial g(t, X_t)}{\partial t} dt + \frac{\partial g(t, X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g(t, X_t)}{\partial x^2} (dX_t)^2$$

For the proof see [6].

Control: The measurable deterministic function $u: \mathbb{R}^n \rightarrow U$ that controls the general solution of any given differential equation so as to give the maximum or minimum value is called control. The set of all controls is usually known as the control region and is often denoted by U that is $U = \{u, u: \mathbb{R}^n \rightarrow U\}$

An impulse control: Is double sequence $v = ((\tau_j, \zeta_j), j = 1, 2, 3, \dots, M)$ where M is random variable taking values in $[0, 1, 2, 3, \dots] \cup \{\infty\}$ consist of:

- 1- Sequence of stopping times $\{\tau_j\}, j = 1, 2, 3, \dots$ such $\tau_j < \infty$ a.s and $\tau_j < \tau_{j+1}$
- 2- Sequence of impulse values $\zeta_1, \zeta_2, \zeta_3, \dots$, such that for each $j = 1, 2, 3, \dots, \zeta_j$ takes values in $Z \subset R$, and ζ_j is measurable with respect to F_{τ_j}

Impulse control v is admissible if

- 1- The corresponding state process $X = X^v$ exists and is unique.
- 2- With probability one, either $M(\omega) < \infty$ or if $M(\omega) = \infty$ then

$$\lim_{j \rightarrow \infty} \tau_j(\omega) = \infty$$

Let $v^k = ((\tau_1, \zeta_1), (\tau_2, \zeta_2), \dots, (\tau_k, \zeta_k))$ to be the first k times, and impulses.

Denote by v^0 to empty impulse control with no intervention on $[0, \infty]$ with $\tau_0 = 0$. Consider the random process $X(t) = x$ at time t , when we give the system an impulse, $\zeta \in Z \cup \{\xi\}$ where Z be a given set not containing ξ , then the result of the impulse is that the process $X(t)$ jumps from x to a new state $X_{t+} = \Gamma(x, \zeta)$, where $\Gamma: R \times (Z \cup \{\xi\}) \rightarrow S \cup \{E\}$ is given function and $S \subseteq R^n$.

Combined random control: A combined random control w is a pair $w = (u, v)$ such that $u \in U$.

Let T be the class of F_- stopping times satisfying $\tau_s \in T$, for all $\tau \in T$, then $\tau \leq \tau_s$ a.s.

Let $V = V_T$ be the set of all impulse controls in the form

$$v = ((\tau_1, \zeta_1), (\tau_2, \zeta_2), \dots,)$$

Define the space $W = W_T$ of admissible combined random controls to be:

$$v = ((\tau_1, \zeta_1), (\tau_2, \zeta_2), \dots,) \in U \times V_T \text{ satisfying} \\ (\tau_k(\omega), \zeta_k) = (\tau_s, \zeta_s), \text{ if } \tau_k(\omega) = \tau_s$$

Reward (cost) function: There is some cost associated with the system, which may depend on the system state itself and on the control used. The reward function is typically expressed as a function $J(x, U)$, representing the expected total cost starting from system state x if control process U is implemented.

Value function: The value function describes the value of the minimum possible cost of the system (or maximum possible reward). It is usually denoted by V and is obtained, for initial state x , by optimizing the cost over all admissible controls.

When we applied a control continuously in time, we will only apply a control action at a sequence of stopping times, i.e., we allow ourselves to give the system impulses at suitably chosen times, this is called an impulse control problem. Such problems are important in a variety of applications, including resource management, inventory management, production planning and economic applications.

As in any stochastic control problem, the goal of an impulse control problem is to find an optimal control and the corresponding value function, for both we have two decisions: whether to intervene or not, and if so what value of control impulse to apply.

MATHEMATICAL FORMULATION OF THE PROBLEM

Now we will discuss the mathematical formulation for the impulse control problem for system driven by Wiener process (which is the standard Brownian motion) and find the required value function, for this assume that $X(t)$ represent the dynamics of the system that described by the following random differential equation.

$$dX(t) = f(X_t)dt + \sigma(X_t)dB_t, \quad X(0) = x \in R^k$$

Where $f: R^k \rightarrow R^k$, $\sigma: R^k \rightarrow R^{n+d}$ are given Lipschitz continuous functions and B_t is d-dimensional Brownian motion on a filtered probability space $(\Omega, F, \{F\}_{t>0}, P)$, $B(0) = 0$ a.s

Suppose that at any time t and any state y we can intervene by giving the system an impulse $\zeta \in Z \subset R^l$.

As we mentioned above the result of the impulse is that the state jumps immediately suppose from $X(t^-) = x$ to $X(t) = \Gamma(x, \zeta)$ where $\Gamma: R^k \times Z \rightarrow R^k$ is given function.

If $v = ((\tau_1, \zeta_1), (\tau_2, \zeta_2), \dots,)$ is applied to the system $\{X(t)\}$ then the value $X^{(v)}(t)$ can be described as follows:

$$\begin{cases} dX^{(v)}(t) = f(X^{(v)}(t))dt + \sigma(X^{(v)}(t))dB_t, & \tau_j < t < \tau_{j+1} \leq T^* \\ X^{(v)}(\tau_{j+1}) = \Gamma(X^{(v)}(\tau_{j+1}^-), \zeta_{j+1}), & j = 0, 1, 2, \dots, \tau_{j+1} < T^* \end{cases}$$

Where T^* is defined by:

$$T^*(\omega) = \lim_{R \rightarrow \infty} \left(\inf \{ t > 0 ; |X^{(v)}(t)| \geq R \} \right) \leq \infty$$

Let $Q^y = Q^{y,v}$ denote the law of the random process $X^{(v)}$ starting at $X^{(v)}(0) = x$.

Let $S \subset R^k$ be a fixed Borel set such that $S \subset (\bar{S}^0)$, where S^0 denotes the interior of S and (\bar{S}^0) it's closure, we are only interested in the system up to the first exit time from S .

Suppose that the profit / utility rate when the system is in state x is $f(x)$ where $f: S \rightarrow R$ is given function. Denote to the boundary of S by ∂S

Let $g: \partial S \rightarrow R$ be a given bequest function. Suppose also that the utility of Performing an intervention with impulse $\zeta \in Z$ when the system is in state y is $K(y, \zeta)$ where $K: S \times Z \rightarrow R$ is given function.

Define $T = T_y^{(v)}(\omega) = \inf \{ t \in (0, T^*(\omega)); X^{(v)}(t, \omega) \notin S \}$

Let V be a given set of admissible controls which includes the set of impulse controls $v = ((\tau_1, \zeta_1), (\tau_2, \zeta_2), \dots,)$ such that:

$$X^{(v)}(t) \in S, \forall t < T, T^* = \infty, \lim_{j \rightarrow \infty} \tau_j = T \text{ a.s.}, Q^y \forall y \in R^k$$

Assume that:

$$E^y \left[\int_s^T |f(Y(t))| dt \right] < \infty$$

$$E^y \left[\int_s^T |g(Y(T))| \chi_{\{T < \infty\}} \right] < \infty$$

$$E^y \left[\sum_{\tau_j < T} |K(Y^{(v)}(\tau_j^-), \varsigma_j)| \right] < \infty$$

Where E^y denote the expectation with respect to Q^y .

When we applying $v = ((\tau_1, \varsigma_1), (\tau_2, \varsigma_2), \dots,) \in V$ to the system then the total expected profit utility is given by:

$$J^v(y) = E^y \left[\int_s^{\hat{t}} f(X^{(v)}(t)) dt + g(X^{(v)}(t)) \chi_{\{T < \infty\}} + \sum_{\tau_j < T} K(X^{(v)}(\tau_j^-), \varsigma_j) \right] < \infty \quad (1)$$

Define the value function $\Phi(y)$ as:

$$\Phi(y) = \sup \{ J^v(y), v \in V \} = J^{(v^*)}(y) \quad (2)$$

The problem is to solve the optimization problem (2), which means we want to find the value function $\Phi(y)$ and the associated optimal control v .

Equation (2) describe the formulation of the impulse control problem as maximum problem.

The formulation for impulse control problems as minimum problem instate of maximum one is describe by

$$\Phi(y) = \inf \{ J^v(y), v \in V \} = J^{(v^*)}(y) \quad (3)$$

Here we have to deal with cost rather than profits.

APPLICATIONS

To show the importance and the applications of the impulse control problems to real life, we will illustrate some areas where this concept is appear.

-Due to various economic factors, the exchange rate between two currencies (say, the dollar and some foreign currency) fluctuates randomly in time. It is not a good idea, however, to have the exchange rate be too far away from unity. As such, the central bank tries to exert its influence on the exchange rates to keep them in a certain .safe. Target zone. One way in which the central bank can influence the exchange rate is by buying or selling large quantities of foreign currency. The question then becomes, at which points in time the central bank should decide to make a large transaction in foreign currency, and for what amount, in order to keep the exchange rate in the target zone. This is an impulse control problem, see [5]

-Suppose that we are given number of assets in which we can invest our money , and at any time we can transfer some of our money from one of the investments to the others , an impulse control problem is to determine that at which

times we should transfer money, how much money and from where to where we transfer the money so as to maximize the expected total utility at some given time $T > 0$

EXAMPLE

(Optimal forest harvesting). We own a forest which is harvested for lumber. When the forest is planted, it starts off with a (nonrandom) total biomass $x_0 > 0$, as the forest grows, the biomass of the forest grows according to the geometric Brownian motion:

$$dX(t) = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = 0, \quad \mu > 0$$

At some time τ_1 , we can decide to cut the forest and sell the wood; we then replant the forest so that it starts off again with biomass x_0 . The forest can then grow freely until we decide to cut and replant again at time τ_2 , etc. Every time τ we cut the forest, we obtain $X_{\tau-}$ dollars from selling the wood, but we pay a fee proportional to the total biomass $X_{\tau-}$ ($0 \leq \alpha < 1$) for cutting the forest, and a fixed fee $Q > 0$ for replanting the forest to its initial biomass x_0 . When inflation, with rate (discount factor) $\lambda > \mu$ is taken into account, the expected future profit from this operation is given by:

$$E \left[\sum_{j=1}^{\infty} e^{-\lambda \tau_j} \left((1-\alpha) X_{\tau_j-} - Q \right) \right].$$

Our goal is to choose a harvesting strategy $\tau_1, \tau_2, \tau_3, \dots$ which maximizes our expected profit, i.e., we wish to choose an impulse control strategy u^* which minimizes

$$J(u) = E \left[\sum_{j=1}^{\infty} e^{-\lambda \tau_j} (Q - (1-\alpha) X_{\tau_j-}) \right]$$

See [5].

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